EQUALITY OF GRAVER BASES AND UNIVERSAL GRÖBNER BASES OF COLORED PARTITION IDENTITIES

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Abstract. Associated to any vector configuration A is a toric ideal encoded by vectors in the kernel of A. Each toric ideal has two special generating sets: the universal Gröbner basis and the Graver basis. While the former is generally a proper subset of the latter, there are cases for which the two sets coincide. The most prominent examples among them are toric ideals of unimodular matrices. Equality of universal Gröbner basis and Graver basis is a combinatorial property of the toric ideal (or, of the defining matrix), providing interesting information about ideals of higher Lawrence liftings of a matrix. Nonetheless, a general classification of all matrices for which both sets agree is far from known. We contribute to this task by identifying all cases with equality within two families of matrices; namely, those defining rational normal scrolls and those encoding homogeneous primitive colored partition identities.

1. Introduction

A vector configuration $A \in \mathbb{Z}^{d\times n}$ represents a toric ideal $\mathrm{I_A} \subseteq k[x_1,\dots,x_n]$ in the following way: For a nonnegative vector $\mathbf{u}^+$, let $\mathbf{x}^{\mathbf{u}^+}$ denote the monomial $x_1^{\mathbf{u}_1^+}\cdots x_n^{\mathbf{u}_n^+}$. A vector $\mathbf{u}$ in the lattice $\ker A$ corresponds to a binomial $\mathbf{x}^{\mathbf{u}^+}-\mathbf{x}^{\mathbf{u}^-}$ after writing $\mathbf{u}=\mathbf{u}^+-\mathbf{u}^-$ such that both $\mathbf{u}^+$ and $\mathbf{u}^-$ have only nonnegative coordinates and disjoint support. The toric ideal $\mathrm{I_A}$ associated to A is the set of all binomials arising from the lattice $\ker A$; that is, $\mathrm{I_A}=\langle \mathbf{x}^{\mathbf{u}^+}-\mathbf{x}^{\mathbf{u}^-} : A\mathbf{u}=\mathbf{0}\rangle$.

The ideal $\mathrm{I_A}$ has finitely many reduced Gröbner bases (e.g., see [14, Chp1]). Denote by $\mathrm{UGB}(A)$ their union, the (minimal) *universal Gröbner basis* of $\mathrm{I_A}$. In general, $\mathrm{UGB}(A)$ is properly contained in the *Graver basis* $\mathcal{G}(A)$, consisting of all binomials $\mathbf{x}^{\mathbf{u}^+}-\mathbf{x}^{\mathbf{u}^-}\in \mathrm{I_A}$ for which there is no other binomial $\mathbf{x}^{\mathbf{v}^+}-\mathbf{x}^{\mathbf{v}^-}\in\mathrm{I_A}$ such that $\mathbf{x}^{\mathbf{v}^+}$ divides $\mathbf{x}^{\mathbf{u}^+}$ and $\mathbf{x}^{\mathbf{v}^-}$ divides $\mathbf{x}^{\mathbf{u}^-}$. Such binomials are called *primitive*. Toric ideals, their generating sets and Gröbner bases play a prominent role in many different areas, such as algebraic geometry, commutative algebra, graph theory, integer programming, and algebraic statistics [6], [16], [14], [4].

Universal Gröbner basis and Graver basis rarely coincide. It is known that they agree for some special toric ideals, including toric ideals of unimodular matrices [13, 15]. However, a general classification of all matrices for which both sets agree is not known. We classify such matrices within two infinite families of interest, one of which defines a classical family of projective varieties.

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Equality of the two bases provides information about higher Lawrence liftings of A, as discussed in [8]. To be more specific, for any $N \in \mathbb{Z}_{>0}$, consider the N-fold matrix

$$[A,B]^{(N)} := \begin{pmatrix} B & B & \cdots & B \\ A & 0 & & 0 \\ 0 & A & & 0 \\ & & \ddots & \\ 0 & 0 & & A \end{pmatrix}$$

associated to integer matrices A and B of suitable dimensions. For $\mathbf{u} = \left(\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(N)}\right) \in \ker\left([A,B]^{(N)}\right)$ we call $\left|i : \mathbf{u}^{(i)} \neq \mathbf{0}\right|$ its *type*. A surprising property of N-fold matrices is that, for all N, the maximum type of an element in $\mathcal{G}([A,B]^{(N)})$ is bounded by some number $g(A,B)$ *not depending* on N. Analogously to this so-called *Graver complexity* $g(A,B)$, one can define the (universal) *Gröbner complexity* $u(A,B)$ as the maximum type of an element in $\mathrm{UGB}([A,B]^{(N)})$ over all N. As $\mathrm{UGB}([A,B]^{(N)}) \subseteq \mathcal{G}([A,B]^{(N)})$, we have $u(A,B) \leq g(A,B)$. The main result of [8] is that if $\mathrm{UGB}(A) = \mathcal{G}(A)$, then $u(A,B) = g(A,B)$ for *all* integer matrices B of suitable dimensions. Hence, we have the following sequence of implications:

$$A \text{ is unimodular } \Rightarrow \mathrm{UGB}(A) = \mathcal{G}(A) \Rightarrow u(A,B) = g(A,B) \text{ for all suitable matrices } B.$$

It is known that the converse of each of these two implications is false. For example, we present infinitely many non-unimodular matrices A with $\mathrm{UGB}(A) = \mathcal{G}(A)$. That the converse of the second implication does not hold has been shown in [8] for matrices of the form $A = \left(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 0 & a & b & a+b \end{smallmatrix}\right)$, for which $u(A, I_4) = g(A, I_4)$, while $\mathrm{UGB}(A) \subsetneq \mathcal{G}(A)$.

Let us define the two families we study. Note that the structure of the matrices resembles N-fold, a fact which we will exploit in our proofs. Given a partition $n_1 \geq n_2 \geq \cdots \geq n_c$ of a positive integer n, define

$$A_{S(n_1-1,\ldots,n_c-1)} := \begin{bmatrix} 1 & 2 & \cdots & n_1 & 1 & \cdots & n_2 & \cdots & 1 & \cdots & n_c \\ 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & & & \ddots & & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 1 \end{bmatrix}$$

and

$$A_{H(n_1,\ldots,n_c)} := \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & 1 & \cdots & 1 \\ 1 & 2 & \cdots & n_1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & n_2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & & & \ddots & & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & n_c \end{bmatrix}.$$

The toric ideal associated to the first matrix is the defining ideal of the c-dimensional *rational normal scroll* $S := S(n_1 - 1, \ldots, n_c - 1)$ in $\mathbb{P}^{n-1}$; see Lemma 2.1 in [12]. In fact, the scroll S is the projective variety whose defining ideal is generated by the 2×2 minors of the matrix $M = [M_{n_1}|\cdots|M_{n_c}]$ of indeterminates, where

$$M_{n_j} = \begin{bmatrix} x_{j,1} & \cdots & x_{j,n_j-1} \\ x_{j,2} & \cdots & x_{j,n_j} \end{bmatrix}.$$

Eisenbud and Harris [5] survey the geometry of these scrolls. In particular, they belong to the family of nondegenerate projective varieties of minimum possible degree: just one more than

the codimension. The case $c = 1$ represents the *rational normal curve* in $\mathbb{P}^{n-1}$. An important combinatorial feature of this family is that the binomials in the ideal of any rational normal scroll encode color-homogeneous colored partition identities, as defined in [12]. For a precise definition, see Section 3.

The second family of matrices $A_{H(n_1,\ldots,n_c)}$ represents toric ideals whose binomials encode homogeneous (but not necessarily color-homogeneous) colored partition identities. Here, in contrast to $A_{S(n_1-1,\ldots,n_c-1)}$, the positions of the incidence vectors of the parts of the partition and the vectors $(1, 2, \ldots, n_i)$ are interchanged. This effectively removes the requirement for homogeneity of the ideal with respect to the rows of the scroll matrix containing the incidence vectors.

Within these two families of matrices, we classify those for which the universal Gröbner and Graver bases coincide. To state the classification, define a *dominance* partial order on partitions: $(n_1, \ldots, n_c) \preceq (m_1, \ldots, m_d)$ if $c \leq d$ and $n_j \leq m_j$ for $j = 1, \ldots, c$. Naturally, this induces a partial order on $S(n_1, \ldots, n_c)$ and on $H(n_1, \ldots, n_c)$: we say that $S(m_1, \ldots, m_d)$ *dominates* $S(n_1, \ldots, n_c)$ if $(n_1, \ldots, n_c) \preceq (m_1, \ldots, m_d)$. Analogously, we say that $H(m_1, \ldots, m_d)$ *dominates* $H(n_1, \ldots, n_c)$ if $(n_1, \ldots, n_c) \preceq (m_1, \ldots, m_d)$.

Our main result is the following.

Theorem 1. *The universal Gröbner basis of $S = S(n_1 - 1, \ldots, n_c - 1)$ does* not *equal its Graver basis if and only if S dominates $S(6)$, $S(5, 4)$, or $S(4, 3, 2)$.*

The universal Gröbner basis of $H = H(n_1, \ldots, n_c)$ does not *equal its Graver basis if and only if H dominates $H(7)$, $H(6, 2)$, or $H(4, 3)$.*

This result has some interesting consequences. First, it answers a question posed in [8], showing that N-fold matrices that preserve the Gröbner and Graver complexities do not preserve equality of universal Gröbner and Graver bases. Namely, there exist integer matrices A and B of appropriate dimensions satisfying $\mathrm{UGB}(A) = \mathcal{G}(A)$, and hence $u(A, B) = g(A, B)$, but that still satisfy $\mathrm{UGB}([A, B]^{(N)}) \subsetneq \mathcal{G}([A, B]^{(N)})$ for all $N > 1$. One such example is given by $S(5)$, that is, $A = (1, 1, 1, 1, 1, 1)$ and $B = (1, 2, 3, 4, 5, 6)$.

Secondly, let us point out that $\mathrm{UGB}(A) = \mathcal{G}(A)$ seems to be a purely *combinatorial* property of A. It is *not* related to the sequence of algebraic properties

$$A \text{ is unimodular } \Rightarrow A \text{ is compressed } \Rightarrow K\text{-algebra } K[A] \text{ is normal (for some field } K),$$

as discussed in [9]. Note that the first family of matrices is not compressed. In addition, among the matrices within the two families that satisfy $\mathrm{UGB}(A) = \mathcal{G}(A)$, one set has this normality property, while the other one doesn't. It is well-known that semigroup algebras $K[A]$ of rational normal scrolls are normal (this follows, for example, by Proposition 13.5 in [14], since every scroll has a squarefree initial ideal). On the other hand, the second family of matrices does not have this property:

Lemma 2. $K[H] := K[H(n_1, \ldots, n_c)]$ *is normal if and only if H does not dominate $H(2, 2)$.*

Proof. Let $H(2,2) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$. Then $K[H(2, 2)]$ is not normal by Lemma 6.1 in [9], since the binomial $\mathbf{x}_1^2\mathbf{x}_4 - \mathbf{x}_2\mathbf{x}_3^2 \in I_{H(2,2)}$ is an indispensible binomial. As we find a similar binomial (after suitable index permutation) in the toric ideal of any H dominating $H(2, 2)$, one implication of the claim follows. It remains to show that if H does not dominate $H(2, 2)$

then K-algebra of H is normal. But then H is dominated by $H(k,1,\ldots,1)$ and hence the corresponding K-algebra is isomorphic to the K-algebra of $S(k-1)$ which is normal. □

As we will see in the following section (in particular, Corollary 5), Theorem 1 gives the classification also for certain submatrices of the matrices we consider here. It is not clear whether, in case both sets do not coincide, the ratios $|\mathrm{UGB}(A_S)|\,/\,|\mathcal{G}(A_S)|$ and $|\mathrm{UGB}(A_H)|\,/\,|\mathcal{G}(A_H)|$ tend to 0 as S and H increase in the dominance ordering. Moreover, it is an interesting open question whether the special structure of our matrices implies that there are only finitely many fundamental counterexamples to equality of universal Gröbner basis and Graver basis, from which all other counterexamples can be derived.

In particular, such results would provide some insight into the complexity of the *Gröbner fan* of varieties of minimal degree. Namely, each reduced Gröbner basis determines a cone in the Gröbner fan of the ideal $\mathrm{I_A}$. A part of the Gröbner fan of rational normal curves is understood: in [1], Conca, De Negri and Rossi determine explicitly all the initial ideals of $I_{S(n)}$ that are Cohen-Macaulay. To determine the rest of the fan for such a curve or for a general scroll remains an open problem. One approach to this problem is to first understand an approximation of the set $\mathrm{UGB}(A)$. Since the Graver basis equals the universal Gröbner basis for only a small set of scrolls, understanding the primitive binomials that do not belong to $\mathrm{UGB}(A)$ is of interest.

2. Preliminaries

Universal Gröbner bases and Graver bases have nice geometric properties that allow us to enumerate the dominance-minimal cases where equality fails. In this section we recall known results fundamental to our problem.

For any $\mathbf{b}\in\mathbb{Z}^d$, the polytope $P^I_{\mathbf{b}} := \mathrm{conv}(\{\mathbf{u} : A\mathbf{u}=\mathbf{b}, \mathbf{u}\in\mathbb{Z}^n_+\})$ is called the *fiber* of $\mathbf{b}$. With this, we can characterize the elements in $\mathrm{UGB}(A)$.

Proposition 3. [14] *A integer vector $\mathbf{u}^+-\mathbf{u}^-$ (with $\mathbf{u}^+,\mathbf{u}^-\geq\mathbf{0}$) lies in* $\mathrm{UGB}(A)$ *if and only if the line segment $[\mathbf{u}^+,\mathbf{u}^-]$ is an edge of the fiber $P^I_{A\mathbf{u}^+}$ and contains no lattice points other than its endpoints.*

This proposition allows us to check computationally whether $\mathrm{UGB}(A)=\mathcal{G}(A)$ for any given matrix A as follows: First, we compute the Graver basis $\mathcal{G}(A)$; then, we test the statement in Proposition 3 for every $\mathbf{u}\in\mathcal{G}(A)$. To check this condition, we first enumerate the set V of all lattice points in the polytope $P_{A\mathbf{u}^+}$ and then check whether $\mathbf{u}$ is an edge of this polytope that contains no interior lattice point. This can be done by first testing whether $\mathbf{u}^+$ and $\mathbf{u}^-$ are vertices of $P_{A\mathbf{u}^+}$ (that is, they are not convex combination of $V\setminus\{\mathbf{u}^-\}$ and of $V\setminus\{\mathbf{u}^+\}$, respectively). If $\mathbf{u}^+$ and $\mathbf{u}^-$ are vertices of $P_{A\mathbf{u}^+}$, then $\mathbf{u}$ is an edge with no integer point in its interior if and only if there does not exist a decomposition

$$\mathbf{u}=\mathbf{u}^+-\mathbf{u}^-=\sum_{\mathbf{v}\in V\setminus\{\mathbf{u}^+,\mathbf{u}^-\}}\lambda_{\mathbf{v}}(\mathbf{v}-\mathbf{u}^-)$$

with non-negative real coefficients $\lambda_{\mathbf{v}}$. This feasibility problem can be decided by any code that solves linear programs. We applied the commercial solver CPLEX [2].

For small scrolls, the universal Gröbner basis can also be computed by the software package Gfan [11]. However, once we reached 9 variables, the program failed to finish this computation

due to its complexity. Even when the universal Gröbner basis itself is not especially large, each binomial occurs in many different Gröbner bases, each one of which must be enumerated in order to compute the Gröbner fan, as Gfan does. In contrast, 4ti2 [7] is very quick in calculating the Graver basis for our two families of matrices. We then use Proposition 3 to extract the universal Gröbner basis from the Graver basis set. Such an approach was necessary to make the computations feasible for reasonably large examples.

Besides this algorithmic test, Proposition 3 also allows us to show a well-known and very useful fact; namely, that certain projections preserve elements in the universal Gröbner and Graver bases:

Corollary 4. *Suppose* $\mathbf{u} \in \ker_{\mathbb{Z}} A$ *and* $\sigma \subseteq [n]$ *is such that* $\mathbf{u}_i = 0$ *for* $i \notin \sigma$. *Let* A_σ *be the submatrix of* A *of columns indexed by* σ *and* $\mathbf{u}_\sigma$ *be the projection of* $\mathbf{u}$ *to* $\mathbb{R}^\sigma$. *Then:*

(a) $\mathbf{u} \in \mathrm{UGB}(A)$ *if and only if* $\mathbf{u}_\sigma \in \mathrm{UGB}(A_\sigma)$.
(b) $\mathbf{u} \in \mathcal{G}(A)$ *if and only if* $\mathbf{u}_\sigma \in \mathcal{G}(A_\sigma)$.

Proof. To prove claim (a), observe that since the hyperplanes $x_i = 0$ $(i \notin \sigma)$ do not pass through the interior of $P_{A\mathbf{u}^+}$, the polytope $P_{A_\sigma \mathbf{u}_\sigma^+}$ is a face of $P_{A\mathbf{u}^+}$. It follows that the segment $[\mathbf{u}^+, \mathbf{u}^-]$ is an edge of $P_{A_\sigma \mathbf{u}_\sigma^+}$ if and only if it is an edge of $P_{A\mathbf{u}^+}$.

To prove claim (b), simply observe that the minimality property of $\mathbf{x}^{\mathbf{v}^+} - \mathbf{x}^{\mathbf{v}^-} \in \mathrm{I_A}$ is the same for both matrices A and A_σ, as the variables indexed by elements of σ do not appear in either case. $\square$

Note that Corollary 4 immediately implies that equality of the two sets is inherited by submatrices:

Corollary 5. *Let* A_σ *be obtained from* A *by first choosing the submatrix of* A *consisting of the columns indexed by* $\sigma \subseteq [n]$ *and by then eliminating some or all of the redundant rows. Then* $\mathrm{UGB}(A) = \mathcal{G}(A)$ *implies* $\mathrm{UGB}(A_\sigma) = \mathcal{G}(A_\sigma)$.

The dominance order on partitions provides the following simple consequence of Corollary 5 (See also Proposition 4.13 in [14]).

Proposition 6. *Suppose* $(n_1, \dots n_c) \prec (m_1, \dots, m_d)$. *Then* $\mathrm{UGB}\left(A_{S(m_1,\dots,m_d)}\right) = \mathcal{G}\left(A_{S(m_1,\dots,m_d)}\right)$ *implies* $\mathrm{UGB}\left(A_{S(n_1,\dots,n_c)}\right) = \mathcal{G}\left(A_{S(n_1,\dots,n_c)}\right)$. *Similarly,* $\mathrm{UGB}\left(A_{H(m_1,\dots,m_d)}\right) = \mathcal{G}\left(A_{H(m_1,\dots,m_d)}\right)$ *implies* $\mathrm{UGB}\left(A_{H(n_1,\dots,n_c)}\right) = \mathcal{G}\left(A_{H(n_1,\dots,n_c)}\right)$.

In particular, this allows us to solve our classification problem by listing only the dominance-minimal matrices for which equality of the universal Gröbner basis and Graver basis does *not* hold.

3. Partitions, Graver Complexity and the Proof of Theorem 1

The toric ideals in the two families we are studying have a nice combinatorial interpretation. The binomials $x_{a_1}x_{a_2}\cdots x_{a_k} - x_{b_1}x_{b_2}\cdots x_{b_k}$ in the ideal $I_{S(n_1-1)}$ of a rational normal curve encode *homogeneous partition identities*:

$$a_1 + \cdots + a_k = b_1 + \cdots + b_k$$

where $a_1, \ldots, a_k, b_1, \ldots, b_k$ are positive integers, not necessarily distinct [3]. Similarly, the binomials in the ideal I_S of a rational normal scroll encode *color-homogeneous colored partition identities* (color-homogeneous cpi's) [12]:

$$a_{1,1} + \cdots + a_{1,k_1} + \cdots + a_{c,1} + \cdots + a_{c,k_c} = b_{1,1} + \cdots + b_{1,k_1} + \cdots + b_{c,1} + \cdots + b_{c,k_c}.$$

For example,

$$1_1 + 5_1 + 1_2 + 5_2 = 2_1 + 6_1 + 2_2 + 2_2$$

is a color-homogeneous cpi encoded by the binomial

$$x_{1,1}x_{1,5}x_{2,1}x_{2,5} - x_{1,2}x_{1,6}x_{2,2}^2,$$

while

$$1_1 + 5_1 + 1_2 = 3_1 + 1_2 + 3_2$$

is a cpi that is homogeneous, but not color-homogeneous, and is encoded by the binomial

$$x_{1,1}x_{1,5}x_{2,1} - x_{1,3}x_{2,1}x_{2,3}.$$

Among such identities, those with no proper sub-identities are again called *primitive* and are encoded by the elements of $\mathcal{G}(A_S)$ [12]. The analogous statement is true for $\mathcal{G}(A_H)$. This attractive characterization of the Graver bases makes it especially useful to classify which scrolls have the property that the universal Gröbner basis is equal to the Graver basis.

Let us now start collecting the ingredients for the proof of our main result.

Lemma 7. *The universal Gröbner basis and the Graver basis are not the same for the defining matrices of* $S(6)$, $S(5,4)$, $S(4,3,2)$, $H(7)$, $H(6,2)$, *and* $H(4,3)$.

Proof. The defining matrix of $S(6)$ is

$$A_{S(6)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Consider the vector $\mathbf{g} = (1,-1,1,-1,-1,0,1) \in \ker(A_{S(6)})$ and the three vectors $\mathbf{u}_1 = (1,0,0,0,2,0,0)$, $\mathbf{u}_2 = (0,2,0,0,0,0,1)$, $\mathbf{u}_3 = (0,0,1,2,0,0,0)$ in the fiber $\{\mathbf{u} : A_{S(6)}\mathbf{u} = A_{S(6)}\mathbf{g}^+, \mathbf{u} \in \mathbb{Z}_+^7\}$ of $\mathbf{g}$. Notice that

$$\mathbf{g} = \mathbf{g}^+ - \mathbf{g}^- = 1 \cdot (\mathbf{u}_1 - \mathbf{g}^-) + 1 \cdot (\mathbf{u}_2 - \mathbf{g}^-) + 1 \cdot (\mathbf{u}_3 - \mathbf{g}^-)$$

and thus $\mathbf{g}$ is not an edge of $P^I_{\mathbf{g}^+}$. Consequently, $\mathbf{x}^{\mathbf{g}^+} - \mathbf{x}^{\mathbf{g}^-} \notin \mathrm{UGB}(A_{S(6)})$. Yet $\mathbf{g}$ represents the homogeneous primitive partition identity

$$1 + 3 + 7 = 2 + 4 + 5$$

and thus $\mathbf{x}^{\mathbf{g}^+} - \mathbf{x}^{\mathbf{g}^-} \in \mathcal{G}(A_{S(6)}) \setminus \mathrm{UGB}(A_{S(6)})$.

The defining matrix of $S(5,4)$ is

$$A_{S(5,4)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Consider $\mathbf{g} = (1,-1,0,0,1,-1,1,-2,0,0,1) \in \ker(A_{S(5,4)})$, $\mathbf{u}_1 = (0,0,0,0,2,0,2,0,0,0,0)$, and $\mathbf{u}_2 = (2,0,0,0,0,0,0,0,0,0,2)$. Note that $\mathbf{u}_1, \mathbf{u}_2 \in \{\mathbf{u} : A_{S(5,4)}\mathbf{u} = A_{S(5,4)}\mathbf{g}^+, \mathbf{u} \in \mathbb{Z}_+^{11}\}$. Then

$$\mathbf{g}^+ = \frac{1}{2} \cdot \mathbf{u}_1 + \frac{1}{2} \cdot \mathbf{u}_2,$$

and thus $\mathbf{g}^+$ is not a vertex of $P^I_{\mathbf{g}^+}$. Consequently, $\mathbf{x}^{\mathbf{g}^+}-\mathbf{x}^{\mathbf{g}^-}\not\in \mathrm{UGB}(A_{S(5,4)})$. Yet $\mathbf{g}$ represents the color-homogeneous primitive partition identity

$$1_1+5_1+1_2+5_2=2_1+6_1+2_2+2_2$$

and thus $\mathbf{x}^{\mathbf{g}^+}-\mathbf{x}^{\mathbf{g}^-}\in \mathcal{G}(A_{S(5,4)})\setminus \mathrm{UGB}(A_{S(5,4)})$.

The defining matrix of $S(4,3,2)$ is

$$A_{S(4,3,2)}=\begin{pmatrix} 1&2&3&4&5&1&2&3&4&1&2&3\\ 1&1&1&1&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&1&1&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&1&1 \end{pmatrix}.$$

Consider the vectors

$$\begin{aligned}
\mathbf{g} &= (-1,0,0,0,1,1,-1,1,-1,1,0,-1)\in\ker(A_{S(4,3,2)}),\\
\mathbf{u}_1 &= (0,0,0,0,1,0,2,0,0,1,0,0),\\
\mathbf{u}_2 &= (1,0,0,0,0,0,0,0,2,1,0,0),\\
\mathbf{u}_3 &= (0,0,0,0,1,2,0,0,0,0,0,1), \text{and}\\
\mathbf{u}_4 &= (1,0,0,0,0,0,0,2,0,0,0,1).
\end{aligned}$$

Note that $\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\in\{\mathbf{u}: A_{S(4,3,2)}\mathbf{u}=A_{S(4,3,2)}\mathbf{g}^+,\mathbf{u}\in\mathbb{Z}^{12}_+\}$. The decomposition

$$\mathbf{g}=\mathbf{g}^+-\mathbf{g}^-=\frac{1}{2}\cdot(\mathbf{u}_1-\mathbf{g}^-)+\frac{1}{2}\cdot(\mathbf{u}_2-\mathbf{g}^-)+\frac{1}{2}\cdot(\mathbf{u}_3-\mathbf{g}^-)+\frac{1}{2}\cdot(\mathbf{u}_4-\mathbf{g}^-),$$

implies that $\mathbf{g}$ is not an edge of $P^I_{\mathbf{g}^+}$ and, consequently, $\mathbf{x}^{\mathbf{g}^+}-\mathbf{x}^{\mathbf{g}^-}\not\in\mathrm{UGB}(A_{S(4,3,2)})$. Yet $\mathbf{g}$ represents the color-homogeneous primitive partition identity

$$5_1+1_2+3_2+1_3=1_1+2_2+4_2+3_3$$

and thus $\mathbf{x}^{\mathbf{g}^+}-\mathbf{x}^{\mathbf{g}^-}\in\mathcal{G}(A_{S(4,3,2)})\setminus\mathrm{UGB}(A_{S(4,3,2)})$.

The defining matrix of $H(7)$ is the same as of $S(6)$ (up to a swap of rows). Thus the toric ideal corresponding to $H(7)$ is the same as for $S(6)$ and the counterexample for $S(6)$ applies here, too.

The defining matrix of $H(6,2)$ is

$$A_{H(6,2)}=\begin{pmatrix} 1&1&1&1&1&1&1&1\\ 1&2&3&4&5&6&0&0\\ 0&0&0&0&0&0&1&2 \end{pmatrix}.$$

Consider $\mathbf{g}=(-1,1,-1,-1,0,1,2,-1)\in\ker(A_{H(6,2)})$, $\mathbf{u}_1=(0,0,0,2,2,0,0,0)$, $\mathbf{u}_2=(2,0,0,0,0,1,0,1)$, and $\mathbf{u}_3=(0,1,2,0,0,0,1,0,0)$. Note that $\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\in\{\mathbf{u}: A_{H(6,2)}\mathbf{u}=A_{H(6,2)}\mathbf{g}^+,\mathbf{u}\in\mathbb{Z}^8_+\}$. Again, $\mathbf{g}$ is not an edge of $P^I_{\mathbf{g}^+}$, since

$$\mathbf{g}=\mathbf{g}^+-\mathbf{g}^-=1\cdot(\mathbf{u}_1-\mathbf{g}^-)+1\cdot(\mathbf{u}_2-\mathbf{g}^-)+1\cdot(\mathbf{u}_3-\mathbf{g}^-).$$

Consequently, $\mathbf{x}^{\mathbf{g}^+}-\mathbf{x}^{\mathbf{g}^-}\not\in\mathrm{UGB}(A_{H(6,2)})$. Yet $\mathbf{g}$ represents the homogeneous primitive colored partition identity

$$2_1+6_1+1_2+1_2=1_1+3_1+4_1+2_2$$

and thus $\mathbf{x}^{\mathbf{g}^+}-\mathbf{x}^{\mathbf{g}^-}\in\mathcal{G}(A_{H(6,2)})\setminus\mathrm{UGB}(A_{H(6,2)})$.

The defining matrix of $H(4,3)$ is

$$A_{H(4,3)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 \end{pmatrix}.$$

Consider $\mathbf{g} = (1,2,1,-2,-3,0,1) \in \ker(A_{H(4,3)})$, $\mathbf{u}_1 = (2,0,2,0,0,0,1)$, and $\mathbf{u}_2 = (0,4,0,0,0,0,1)$. Note that $\mathbf{u}_1, \mathbf{u}_2 \in \{\mathbf{u} : A_{H(4,3)}\mathbf{u} = A_{H(4,3)}\mathbf{g}^+, \mathbf{u} \in \mathbb{Z}^7_+\}$. This time, $\mathbf{g}^+$ is not a vertex of $P^I_{\mathbf{g}^+}$ since

$$\mathbf{g}^+ = \frac{1}{2}\cdot \mathbf{u}_1 + \frac{1}{2}\cdot \mathbf{u}_2.$$

On the other hand, $\mathbf{g}$ represents the homogeneous primitive colored partition identity

$$1_1 + 2_1 + 2_1 + 3_1 + 3_2 = 4_1 + 4_1 + 1_2 + 1_2 + 1_2,$$

and thus $\mathbf{x}^{\mathbf{g}^+} - \mathbf{x}^{\mathbf{g}^-} \in \mathcal{G}(A_{H(4,3)}) \setminus \mathrm{UGB}(A_{H(4,3)})$. $\square$

Consider the families $S_{c,m-1} := S(m-1, m-1, \ldots, m-1)$ and $H_{c,m} := S(m, m, \ldots, m)$ with c components $m-1$ and m, respectively. The corresponding matrices $A_{c,m-1} := A_{S(m-1,\ldots,m-1)}$ and $B_{c,m} := A_{H(m,\ldots,m)}$ have a special structure in this case: they are c-fold matrices of the form $A_{c,m-1} := [C, D]^{(c)}$ and $B_{c,m} := [D, C]^{(c)}$ with $C = (1,1,\ldots,1)$ and $D = (1,2,\ldots,m)$.

The special structure of these matrices allows us to apply known results about generalized higher Lawrence liftings [10] that the c-fold matrix represents. In particular, this structure implies that, as c grows, the Graver bases of $A_{c,m}$ eventually stabilize (see [10]), in the sense that the type $\left|i : \mathbf{u}^{(i)} \neq \mathbf{0}\right|$ of a vector $\mathbf{u} = \left(\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(c)}\right) \in \mathcal{G}\left(A_{c,m}\right)$ is bounded by a constant $g(C, D)$, the so-called *Graver complexity* of C and D. A similar bound $g(D, C)$ exists for the type of the Graver basis elements of $B_{c,m}$. Note, however, that generally $g(C, D) \neq g(D, C)$.

Lemma 8. *For any fixed m, $C = (1, 1, \ldots, 1)$, and $D = (1, 2, \ldots, m)$, the Graver complexities of the c-fold matrices satisfy $g(C, D) = 2m - 3$ and $g(D, C) \leq 4m - 7$.*

Proof. The Graver complexity $g(C, D)$ can be computed via the algorithm presented in [10]:

$$g(C, D) = \max\{\|\mathbf{v}\|_1 : \mathbf{v} \in \mathcal{G}(D \cdot \mathcal{G}(C))\}.$$

The Graver basis of the $1 \times m$ matrix $C = (1,1,\ldots,1)$ consists of all vectors $\mathbf{e}_i - \mathbf{e}_j$, $1 \leq i \neq j \leq m$. Multiplying these elements by the $1 \times m$ matrix $D = (1,2,3,\ldots,m)$, we conclude that the desired Graver complexity equals the maximum 1-norm among the Graver basis elements of the matrix $(1,2,3,\ldots,m-1)$, which is known to be $2(m-1)-1 = 2m-3$ [3].

Similarly, we can compute $g(D, C)$. The Graver basis elements of $D = (1,2,3,\ldots,m)$ have a maximum 1-norm of $2m-1$. However, as no element in $\mathcal{G}(D)$ has only nonnegative entries, we have that $|C\mathbf{g}| \leq 2m-3$. Hence, $g(D, C)$ is bounded from above by the maximum 1-norm among the Graver basis elements of the matrix $(1,2,3,\ldots,2m-3)$, which is $2(2m-3)-1 = 4m-7$. $\square$

This lemma can be exploited computationally as it bounds the sizes c of $A_{c,m}$ and $B_{c,m}$ for which equality of the universal Gröbner basis and the Graver basis have to be checked. We start with two simple cases.

Lemma 9. *The universal Gröbner basis equals the Graver basis for $A_{c,3}$ and for $B_{c,3}$ for any $c \in \mathbb{Z}_{>0}$.*

Proof. We first verify computationally that the universal Gröbner basis equals the Graver basis for $S(3,3,3,3,3)$ and for $H(3,3,3,3,3)$. Now suppose $\mathbf{g} \in \mathcal{G}(A_{c,3})$ for some $c > 5$. By Lemma 8, the type of any Graver basis element of $A_{c,3}$ is at most $2 \cdot 4 - 3 = 5$, so $\mathbf{g}$ represents a color-homogeneous primitive partition identity with at most five colors. Define $\mathbf{g}'$ by restricting $\mathbf{g}$ to the 20 coordinates that represent these five colors (eliminating only zeros.) Then $\mathbf{g}' \in \mathcal{G}(A_{5,3}) = \mathrm{UGB}(A_{5,3})$. Thus, by applying Corollary 4, we see that $\mathbf{g} \in \mathrm{UGB}(A_{c,3})$.

Similar arguments apply to $B_{c,3}$. Here, the type of any Graver basis element is bounded by $4 \cdot 3 - 7 = 5$, and the result follows from the equality of the universal Gröbner basis and Graver basis for $B_{5,3}$. □

Lemma 10. *For the matrices of $S(5)$, $S(5,3,1,\dots,1)$, $S(5,2,\dots,2)$, $S(4,4,1,\dots,1)$, $H(6)$, and $H(5,2,\dots,2)$, the universal Gröbner basis equals the Graver basis for any number of 1's and 2's, respectively.*

Proof. The cases $S(5)$ and $H(6)$ correspond to the same toric ideal and equality can be verified computationally using Proposition 3. Moreover, we verify computationally that the universal Gröbner basis equals the Graver basis for the following cases: $S(5,3,1,1,1,1,1,1,1)$, $S(5,2,2,2,2,2,2,2,2)$, $S(4,4,1,1,1,1,1)$, and $H(5,2,\dots,2)$ (with 12 components 2). To see that the result of the lemma now follows for any number of 1's and 2's, let us give the arguments for $H(5,2,\dots,2)$. The other cases can be handled analogously.

Let $H = H(5,2,\dots,2)$ with k components 2. If $k \leq 12$, the result follows from the equality for $k = 12$ and Proposition 6. Let $k > 12$ and let $\mathbf{g} \in \mathcal{G}(A_{H(5,2,\dots,2)})$. First observe that $A_{H(5,2,\dots,2)}$ can be obtained from $A_{H(5,5,\dots,5)}$ by deleting certain columns and zero rows thereafter. Thus, $\mathbf{g}$ can be lifted (by only adding components 0) to $\mathbf{g}' \in \mathcal{G}(A_{H(5,5,\dots,5)})$ by Corollary 4. The type of any Graver basis element of $A_{H(5,5,\dots,5)}$ is bounded by $4 \cdot 5 - 7 = 13$. This implies that the type of $\mathbf{g}'$ is at most 13. Therefore, it can be projected (by only removing zero components) to a Graver basis element $\mathbf{g}'' \in \mathcal{G}(A_{H(5,2,\dots,2)})$ for 12 components 2. As we have verified computationally that $\mathrm{UGB}(A_{H(5,2,\dots,2)}) = \mathcal{G}(A_{H(5,2,\dots,2)})$ in this case, we conclude that $\mathbf{g}'' \in \mathrm{UGB}(A_{H(5,2,\dots,2)})$ for 12 components 2. Applying Corollary 4 twice, we conclude that $\mathbf{g} \in \mathrm{UGB}(A_{H(5,2,\dots,2)})$ with $k > 12$ components 2, and thus, $\mathrm{UGB}(A_H) = \mathcal{G}(A_H)$ as claimed. □

Let us finally prove our main theorem.

Proof of Theorem 1.

Suppose that $S = S(n_1 - 1, \dots, n_c - 1)$ is any scroll. If S dominates $S(6)$, $S(5,4)$, or $S(4,3,2)$, then the universal Gröbner basis and Graver basis do not agree by Proposition 6 and Lemma 7. Thus, let us assume that S does not dominate $S(6)$, $S(5,4)$, or $S(4,3,2)$. Then one of the following four cases applies.

Case 1: $n_1 \leq 4$.

Then $S \preceq S_{c,3}$ for some c and thus equality holds by Lemma 9.

Case 2: $n_1 = n_2 = 5$.

Then $n_3 \leq 2$ to avoid dominating $S(4,3,2)$. Thus, we have $S = S(4,4,1,\dots,1)$.

Case 3: $n_1 = 5$ or $6, n_2 \leq 3$.
Then $S \preceq S(5, 2, \ldots, 2)$.

Case 4: $n_1 = 5$ or $6, n_2 = 4$.
Then $n_3 \leq 2$ to avoid dominating $S(4, 3, 2)$. Thus, we have $S \preceq S(5, 3, 1, \ldots, 1)$.

By Proposition 6 and Lemma 10, the universal Gröbner basis and Graver basis coincide for the scrolls S in Cases 2, 3, and 4.

Suppose now that $H = H(n_1, \ldots, n_c)$. If H dominates $H(7)$, $H(6, 2)$, or $H(4, 3)$, then universal Gröbner basis and Graver basis do not agree by Proposition 6 and Lemma 7. Thus, let us assume that H does not dominate $H(7)$, $H(6, 2)$, or $H(4, 3)$. Then one of the following three cases applies.

Case 1: $n_1 \leq 3$.
Then $H \preceq H_{c,3}$ for some c and thus equality holds by Lemma 9.

Case 2: $n_1 = 4$ or 5.
Then $n_2 \leq 2$ to avoid dominating $H(4, 3)$. Then $H \preceq H(5, 2, \ldots, 2)$.

Case 3: $n_1 = 6$.
Then $n_2 \leq 1$ to avoid dominating $H(6, 2)$. But then H has the same toric ideal as $H(6)$.

By Proposition 6 and Lemma 10, the universal Gröbner basis and Graver basis coincide for H in Cases 2 and 3. □

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